The exact roots of the closed-loop characteristic at $K=2$ are at $s=-12.2$, $s=-0.546\pm j1.804$, and $s=-17.35\pm j3.550$, and the dominant pair yields the exact damping ratio of 0.282. The small error in the method, primarily attributable to the influence of other modes, may be somewhat reduced by using both forward and backward perturbations around the natural frequency.

The secondary damping contributed by the other pair of complex poles may sometimes be found by a similar procedure, provided that the mode is well separated from all other modes and that the higher frequency signals are sufficiently large to be measurable without noise interference or overly sophisticated instrumentation systems. In this particular example, the real pole at $s=-12.2$ contributes significant change in phase angle in the spectral region dominated by the other complex poles, thus causing the method to fail.

Conclusions

A simple method for estimating the damping that is present in systems dominated by a single pair of complex conjugate poles has been illustrated. The method constitutes an addition to the tool kit of the dynamic systems experimentalist by providing a relatively straightforward means for the quantitative determination of damping by measurement.

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Use of the Work-Energy Rate Principle for Designing Feedback Control Laws

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Introduction

IT is often felt that control designers do not exploit the principles of analytical dynamics in designing feedback control laws for physical systems. It is shown here that for a class of physical systems, feedback control laws are naturally obtained from the system dynamics by using the Work-Energy Rate Principle. It will be seen that the method applies to a wide variety of linear/nonlinear and discrete/continuous systems.

One of the powerful methods of designing feedback control laws for nonlinear systems is based on Lyapunov's second stability theorem.[1] Energy-type quadratic functions are usually used[2-6] in the application of the Lyapunov stability theorem. The time derivative of the Lyapunov function is obtained by substituting the system equations to eliminate acceleration terms. This process is often tedious and involves integration by parts for distributed parameter systems such as flexible spacecraft. If the time derivative of the Lyapunov function can be obtained without substituting the equations of motion, then the control design efforts will be reduced drastically. This is the motivation behind this Note. The expression for the rate of change of system energy is formulated using the Virtual Work Principle,[7,8] and two examples are given to show the power of this method.

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Work-Energy Rate Principle

When the system is composed of N particles, it is configured by N physical position vectors $\boldsymbol{x}_k$ measured relative to an inertial frame or $n(n\leq 3N)$ generalized coordinates q_i. The coordinate transformations between the physical and the generalized coordinates are given by

$$x_k = x_k(q_1,q_2,\ldots,q_n,t), \qquad (k=1,2,\ldots,N) \tag{1}$$

and their time derivatives are related by

$$\dot{x}_k = \sum_{i=1}^{n} \frac{\partial x_k}{\partial q_i}\dot{q}_i + \frac{\partial x_k}{\partial t} \tag{2}$$

The forces exerted on the kth particle can be grouped as the applied force $\boldsymbol{F}_k$ and the constraint force $\boldsymbol{R}_k$. When the constraints are workless, the Virtual Work Principle is stated as

$$\sum_{k=1}^{N} (F_k - m_k\ddot{x}_k)^T \delta x_k = 0 \tag{3}$$

where δx_k indicates the kth virtual displacement vector. Using the generalized virtual displacement identity and generalized force definition

$$\delta x_k = \sum_{i=1}^{n}\left(\frac{\partial x_k}{\partial q_i}\right)\delta q_i \tag{4}$$

$$Q_i = \sum_{k=1}^{N} F_k^T \frac{\partial x_k}{\partial q_i} \tag{5}$$

Eq. (3) can be rewritten as

$$\sum_{i=1}^{n}\left(Q_i - \sum_{k=1}^{N} m_k\ddot{x}_k^T\frac{\partial x_k}{\partial q_i}\right)\delta q_i = 0 \tag{6}$$

For the general case, suppose that the system has m nonholonomic constraints given as

$$\sum_{i=1}^{n} a_{ji}\delta q_i = 0 \qquad (j=1,2,\ldots,m)$$

then by using Lagrange multipliers λ_j, Eq. (6) can be written as

$$\sum_{i=1}^{n}\left(Q_i + \sum_{j=1}^{m}\lambda_j a_{ji} - \sum_{k=1}^{N} m_k\ddot{x}_k^T\frac{\partial x_k}{\partial q_i}\right)\delta q_i = 0 \tag{7}$$

and we get

$$\sum_{k=1}^{N} m_k\ddot{x}_k^T\frac{\partial x_k}{\partial q_i} = Q_i + \sum_{j=1}^{m}\lambda_j a_{ji} \qquad (i=1,2,\ldots,n) \tag{8}$$

Multiplying $\dot{q}_i$ on both sides of Eq. (8) and summing for all i leads to an interesting result: the left side of Eq. (8) becomes

$$\sum_{i=1}^{n}\sum_{k=1}^{N} m_k \ddot{x}_k^T \frac{\partial x_k}{\partial q_i}\dot{q}_i = \sum_{k=1}^{N} m_k \ddot{x}_k^T \sum_{i=1}^{N} \frac{\partial x_k}{\partial q_i}\dot{q}_i$$

$$= \sum_{k=1}^{N} m_k \ddot{x}_k^T \dot{x}_k - \sum_{k=1}^{N} m_k \ddot{x}_k^T \frac{\partial x_k}{\partial t}$$

$$= \frac{dT}{dt} - \sum_{k=1}^{N} m_k \ddot{x}_k^T \frac{\partial x_k}{\partial t} \tag{9}$$

Thus we obtain

$$\frac{dT}{dt} = \sum_{i=1}^{n} Q_i \dot{q}_i + \sum_{i=1}^{n} C_i \dot{q}_i + \sum_{k=1}^{N} m_k \ddot{x}_k^T \frac{\partial x_k}{\partial t} \tag{10}$$

where

$$C_i \equiv \sum_{j=1}^{m} \lambda_j a_{ji}$$

are the generalized constraint forces. Equation (10) represents the Work-Energy Rate Principle in the generalized coordinate space. It can be stated as, the rate of change of total kinetic energy of the system is equal to the rate of change of work done by applied forces F_k and constraint forces R_k. By substituting Eq. (5) into Eq. (10) and using the facts $F_k + R_k = m_k \ddot{x}_k$ and

$$\sum_{i=1}^{n} C_i \dot{q}_i = \sum_{i=1}^{n}\sum_{k=1}^{N} R_k^T \frac{\partial x_k}{\partial q_i}\dot{q}_i \tag{11}$$

we obtain

$$\frac{dT}{dt} = \sum_{k=1}^{N} F_k^T \dot{x}_k + \sum_{k=1}^{N} R_k^T \dot{x}_k \tag{12}$$

This equation also represents the Work-Energy Rate Principle in the physical coordinates, which can be obtained directly from the kinetic energy expression.

Because Eqs. (10) and (12) include the constraint forces, their direct applications to practical problems are difficult. However, if constraints are holonomic and δq_i are independent, then from the workless condition

$$\sum_{k=1}^{N} R_k^T \delta x_k = \sum_{i=1}^{n}\sum_{k=1}^{N} R_k^T \frac{\partial x_k}{\partial q_i}\delta q_i = 0 \tag{13}$$

we can see that

$$\sum_{k=1}^{N} R_k^T \frac{\partial x_k}{\partial q_i} = 0 \tag{14}$$

Therefore, Eqs. (10) and (12) can be written, respectively, as

$$\frac{dT}{dt} = \sum_{i=1}^{n} Q_i \dot{q}_i + \sum_{k=1}^{N} m_k \ddot{x}_k^T \frac{\partial x_k}{\partial t} \tag{15}$$

$$\frac{dT}{dt} = \sum_{k=1}^{N} F_k^T \dot{x}_k + \sum_{k=1}^{N} R_k^T \frac{\partial x_k}{\partial t} \tag{16}$$

which represent the Work-Energy Rate Principle for holonomic systems.[8] For scleronomic (no explicit time dependence) systems, Eqs. (15) and (16) can be reduced further as

$$\frac{dT}{dt} = \sum_{i=1}^{n} Q_i \dot{q}_i \tag{17}$$

$$\frac{dT}{dt} = \sum_{k=1}^{N} F_k^T \dot{x}_k \tag{18}$$

The generalized force can be written as[7]

$$Q_i = -\frac{\partial V}{\partial q_i} - \frac{\partial F_d}{\partial \dot{q}_i} + Q_i^a \tag{19}$$

where F_d represents the quadratic Rayleigh's dissipation function, V is the potential energy, and Q_i^a are due to other external nonconservative forces. Then Eq. (17) can be written as

$$\frac{d}{dt}(T+V) = -2F_d + \sum_{i=1}^{n} Q_i^a \dot{q}_i \tag{20}$$

Note that when there is no nonconservative applied force the rate of change of system total energy is $-2F_d$ (see Ref. 7). Equation (20) will be shown useful in the design of feedback control laws in many applications. We confine ourselves to the scleronomic and holonomic cases in the examples that follow.

Applications

Two examples involving the design of feedback control laws for spacecraft maneuvers are presented.

Rigid Spacecraft Maneuver with Torque Controllers

Consider a rigid spacecraft with three torque devices. The states are represented by Euler parameter vector β and the angular velocity vector ω. The final state is the rest state in which the vehicle frame coincides with the inertial frame $\{\hat{N}\}$. The system kinematic equations are given as

$$\dot{\beta} = \frac{1}{2}\begin{bmatrix} -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \omega \equiv \frac{1}{2} G(\beta)\omega \tag{21}$$

Let the trial Lyapunov function be an energy-type quadratic function:

$$L = T + ½(\beta - \beta_f)^T K_1 (\beta - \beta_f) \tag{22}$$

where K_1 is a symmetric positive definite matrix, T represents the system kinetic energy, and β_f denotes the object state, i.e., $\beta_f = [1 \quad 0 \quad 0 \quad 0]^T$; L is positive definite and radially unbounded. The time derivative of L is

$$\dot{L} = \dot{T} + \omega^T G^T(\beta) K_1 (\beta - \beta_f) \tag{23}$$

Using the Work-Energy Rate Principle [Eq. (17)], $\dot{T}$ can be written as

$$\dot{T} = \omega^T u \tag{24}$$

where u represents the control torque applied on the spacecraft. A feedback control law that makes $\dot{L}$ negative semidefinite can be chosen as

$$u = -K_2\omega - G^T(\beta) K_1 (\beta - \beta_f) \tag{25}$$

where K_2 is a positive definite matrix. The same control law can also be obtained by substituting the dynamic equations of motion into Eq. (33) by following the procedure outlined in Refs. 2 and 3. This simple example introduces the application of the Work-Energy Rate Principle. Its value can be better appreciated when applied to a more complicated problem.

Flexible Spacecraft Maneuver with Multiple Controllers

Consider a flexible spacecraft that has a central rigid hub and four flexible appendages, as shown in Fig. 1. It is equipped with torque controllers on each appendage as well as a central torque controller on the hub. Our objective is to maneuver the

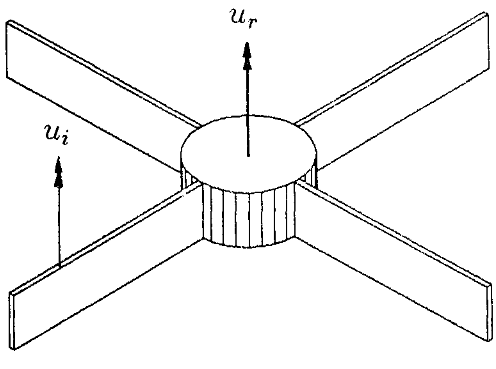

Fig. 1 Flexible spacecraft with controllers.

spacecraft to the rest state and to suppress the bending vibration of the appendages. A single-axis maneuver is considered in this example for simplicity.

Let the candidate Lyapunov function be made up of the total energy and a pseudopotential energy of rotation:

$$L = E + ½ k_\theta(\theta - \theta_f)^2, \qquad k_\theta > 0 \tag{26}$$

where θ represents the present rotational angle and θ_f is the object angle (fixed). From the Work-Energy Rate Principle,

$$\dot{E} = u_r\dot{\theta} + \sum_{i=1}^{n} u_i\left[\dot{\theta} + \frac{\partial}{\partial t}\left(\frac{\partial y_i}{\partial x}\right)\right] \tag{27}$$

where u_r is the central controller torque and u_i is the ith appendage controller torque. Let $\dot{\theta}_i \equiv (\partial/\partial t)[(\partial y_i/\partial x)]$ be the rate of change of slope at the ith appendage controller location. Then, the time derivative of L is

$$\dot{L} = u_r\dot{\theta} + k_\theta(\theta - \theta_f)\dot{\theta} + \sum u_i(\dot{\theta} + \dot{\theta}_i) \tag{28}$$

The hub and appendage control laws can then be obtained as

$$u_r = -K_r\dot{\theta} - k_\theta(\theta - \theta_f) \tag{29}$$

$$u_i = -K_i(\dot{\theta} + \dot{\theta}_i) \tag{30}$$

where k_θ, K_r, and K_i are positive gains. Note that this approach does not need the assumption of small deflections of the appendages and consideration of specific mode shapes. All controllable modes are stabilized and no discretization errors corrupt the stability arguments. Derivations of control laws for the system from the hybrid ordinary/partial differential equations of motion (without using the Work-Energy Rate Principle) can be found in Refs. 4–6.

Conclusions

The Work-Energy Rate Principle in the generalized coordinate space is applied to design feedback control laws for the class of systems that admit total energy as a part of the Lyapunov function. It is shown that the use of this principle reduces the effort required to design feedback control laws for scleronomic and holonomic systems. It is also shown that equations of motion need not be derived to design the feedback control laws and all developments are extensions of the work/energy method and the Principle of Virtual Work. All of the feedback control laws derived in these examples render the closed-loop system asymptotically stable. This method can be applied to rheonomic or nonholonomic systems by properly defining the constraint forces.

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Kane's Equations, Lagrange's Equations, and Virtual Work

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Introduction

THE literature does not appear to contain a derivation of Kane's equations from some first principle. In general, Kane's generalized forces are presented as definitions, from which Kane's equations then follow.[1-4] In this Note, Kane's generalized forces and equations are derived from a first principle—the work-energy form of Newton's second law. Lagrange's equations can also be derived from this same basic form; although it differs conceptually from the usual virtual work (also known as d'Alembert principle) derivations, many of the steps are similar. These parallel derivations clearly show the commonality of Kane's and Lagrange's equations and the occurrence of virtual work-type terms.

The common feature of Lagrange's and Kane's equations is the transformation to generalized coordinates so that system constraint forces are (or can be) eliminated. In the usual derivations of Lagrange's equations, this is attributed to the restricted character of the virtual displacements[5-13]: displacements "for which the virtual work of the forces of constraint vanishes."[5] In Kane's equations, forces of constraint are eliminated if the vector multiplied (dot product) into Newton's law is chosen properly.[2] In some derivations of Lagrange's equations, the vector multiplied into Newton's law is similarly chosen.[11,14] In all cases, explicit time-varying (rheonomic) kinematical terms are discarded, or simply not considered, often without comment. When explanations of virtual displacements and properly chosen vectors are presented, they tend to be convoluted and unenlightening.[2,5-13] The present derivations avoid these ambiguities and explanations and possibly provide increased generality.

Derivations

The derivations are for a holonomic system of p constant-mass particles. Inclusion of rigid bodies is straightforward.

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